FLOW OF LIQUID FROM AN INFINITELY LONG AXIALLY SYMMETRIC CONTAINER

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The flow of liquid from a container limited by infinite plane walls was first studied by Kirchhoff [1].

Trefftz [2] treated the outflow of liquid through a circular orifice in a plane wall. He constructed an integral equation for the velocity potential, which also included a function determining the form of the jet. In order to solve this equation he assumed certain forms of jet.

The author determines the velocity potential by numerical methods, compares it with a known value of the potential at the jet boundary and selects the form of the jet with the smallest difference.

In this article an integro-differential equation is set up also, which is solved by successive approximations; meanwhile the form of the jet is determined in the process of solution.

The scheme analyzed is presented in Fig. 1. A liquid flows from a container of conical form CBB_1C_1, confined by infinite walls BC and B_1C_1, and produces a jet BDD_1B_1. The form of the jet and the velocity distribution along the solid wall are to be determined. Cylindrical coordinates are used for the solution of this problem.

Coordinates of any arbitrary point are indicated as z, r and coordinates of a point at the surface of the container or the jet as z', r'.

Equations representing BC and B_1C_1 are:

$$r' = 1 - z' \operatorname{tg} \beta_1 \qquad (BC)$$
$$r' = -(1 - z' \operatorname{tg} \beta_1) \qquad (B_1C_1) \tag{1}$$

Circles $r = r'$ are considered, located in a plane $z = z'$. Vortices run along every circle. A circulation $\gamma(z')dl'$ is associated with an arc

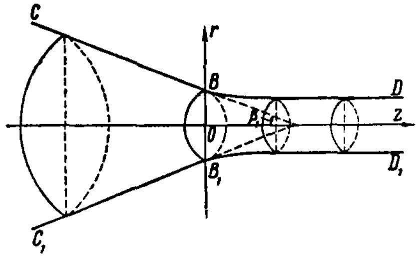

Fig. 1.

element dl' along the boundary. This circulation is assumed to be counterclockwise in the meridional semiplane. Then the stream function is [3]:

$$\psi = \frac{r}{4\pi} \int_{-\infty}^{+\infty} \left[r' \gamma(z') \int_0^{2\pi} \frac{\cos\alpha\, d\alpha}{\rho(r, r', z, \alpha)} \right] dl' \tag{2}$$

where

$$\rho(r\ r', z\ \alpha) = \sqrt{r^2 + r'^2 + (z - z')^2 - 2rr'\cos\alpha}$$

Further

$$v_r = -\frac{1}{r}\frac{\partial\psi}{\partial z}, \qquad v_z = \frac{1}{r}\frac{\partial\psi}{\partial r} \tag{3}$$

Along the entire surface $v_n = 0$, therefore

$$v_r \frac{dr}{dn} + v_z \frac{dz}{dn} = 0$$

since $dr/dn = dz/dl$, $dz/dn = - dr/dl$, therefore

$$v_r dz - v_z dr = 0, \quad \text{or} \quad v_r - v_z \frac{dr}{dz} = 0 \tag{4}$$

At a sufficient distance from the outlet the value of $\gamma(z)$, which corresponds to the tangential velocity component, is inversely proportional to the square of the radius of the container in inifinity upstream. This follows from the discharge continuity condition.

Assume that starting at a point $z = z_1$

$$\gamma(z) = \gamma_1 \left(\frac{r_1}{r}\right)^2 \quad (z \leqslant z_1), \qquad \gamma_1 = \gamma(z_1),\ r_1 = r(z_1) \tag{5}$$

The value $\gamma(z)$ is constant on the surface, and can be taken as unity.

The form of the jet may be assumed practically cylindrical at a certain distance downstream from the outlet, say, from the point $z = z_2$. Equation (2) is presented in following form:

$$\psi = \psi_1 + \psi_2 + \psi_3 \tag{6}$$

where

$$\psi_1 = \frac{r}{4\pi} \int_{-\infty}^{z_1} \left[r'\gamma(z') \int_0^{2\pi} \frac{\cos \alpha d\alpha}{\rho(r, r', z, \alpha)} \right] dl' \tag{7}$$

$$\psi_2 = \frac{r}{4\pi} \int_{z_1}^{z_2} \left[r'\gamma(z') \int_0^{2\pi} \frac{\cos \alpha d\alpha}{\rho(r, r', z, \alpha)} \right] dl' \tag{8}$$

$$\psi_3 = \frac{r}{4\pi} \int_{z_2}^{\infty} \left[r'\gamma(z') \int_0^{2\pi} \frac{\cos \alpha d\alpha}{\rho(r, r', z, \alpha)} \right] dl' \tag{9}$$

Equations (3) after substitution of (6) will be

$$v_r = -\frac{1}{r}\frac{\partial\psi}{\partial z} = -\frac{1}{r}\left(\frac{\partial\psi_1}{\partial z} + \frac{\partial\psi_2}{\partial z} + \frac{\partial\psi_3}{\partial z}\right) = v_{1r} + v_{2r} + v_{3r} \tag{10}$$

$$v_z = \frac{1}{r}\frac{\partial\psi}{\partial r} = \frac{1}{r}\left(\frac{\partial\psi_1}{dr} + \frac{\partial\psi_2}{dr} + \frac{\partial\psi_3}{\partial r}\right) = v_{1z} + v_{2z} + v_{3z}$$

Abbreviations introduced here are obvious. Equation (4) may be presented in the form

$$v_{1r} + v_{2r} + v_{3r} - (v_{1z} + v_{2z} + v_{3z})\frac{dr}{dz} = 0 \tag{11}$$

When the radius of the cylinder is r_0, then

$$\psi_3 = \frac{rr_0}{4\pi} \int_{z_2}^{\infty} \left[\int_0^{2\pi} \frac{\cos \alpha d\alpha}{\rho(r, r_0, z, \alpha)} \right] dz'$$

Zhukovskii [4] has shown that such integrals reduce to elliptic integrals of the first, second and third kind.

Thus, integrating in parts for α, changing the order of integration, and substituting the variable $\alpha = \pi + 2\phi$, we obtain

$$\psi_3 = \frac{rr_0}{4\pi} \int_{z_2}^{\infty} \left[\int_0^{2\pi} \frac{\cos \alpha d\alpha}{\rho(r, r_0, z, \alpha)} \right] dz' = \frac{r^2 r_0^2}{4\pi} \int_{z_2}^{\infty} \left[\int_0^{2\pi} \frac{\sin^2 \alpha d\alpha}{\rho^3(r, r_0, z, \alpha)} \right] dz' =$$

$$= \frac{r^2 r_0^2}{4\pi} \int_0^{2\pi} \left[\int_{z_2}^{\infty} \frac{dz'}{\rho^3(r, r_0, z, \alpha)} \right] \sin^2 \alpha d\alpha =$$

$$= \frac{\lambda\sqrt{rr_0}(z - z_2)}{2\pi n}\left[(1-n)(K_\lambda - J) + \frac{n}{\lambda^2}(K_\lambda - E_\lambda)\right] + \begin{cases} \dfrac{\pi r^2}{4\pi} & \text{for } r < r_0 \\[2mm] \dfrac{\pi r_0^2}{4\pi} & \text{for } r > r_0 \end{cases} \tag{12}$$

Here

$$K_\lambda = \int_0^{\pi/2} \frac{d\varphi}{\sqrt{1-\lambda^2 \sin^2\varphi}}, \qquad E_\lambda = \int_0^{\pi/2} d\varphi \sqrt{1-\lambda^2 \sin^2\varphi}$$

$$J = J(-n, \lambda) = \int_0^{\pi/2} \frac{d\varphi}{(1-n\sin^2\varphi)\sqrt{1-\lambda^2\sin^2\varphi}} \tag{13}$$

$$\lambda = \frac{2\sqrt{rr_0}}{\sqrt{(z-z_2)^2+(r+r_0)^2}}, \qquad n = \frac{4rr_0}{(r+r_0)^2}$$

The formulas (12) substituted into (10) result in

$$v_{3r} = -\frac{\sqrt{rr_0}}{2\pi r\lambda}\left[(2-\lambda^2)K_\lambda - 2E_\lambda\right] \tag{14}$$

$$v_{3z} = \frac{(z-z_2)\lambda}{4\pi\sqrt{rr_0}}\left[\frac{r_0-r}{r_0+r}J + K_\lambda\right] + \begin{cases} \frac{1}{2} \text{ for } r<r_0 \\ 0 \text{ for } r>r_0 \end{cases} \tag{15}$$

Values K_λ and E_λ are to be taken from the table of elliptic integrals. The complete elliptic integral of the third kind J resolves into partial elliptical integrals of the first and second kind by the following formula (4):

$$J(-n, \lambda) = \frac{\sqrt{1-\lambda'^2\sin^2\delta}}{\lambda'^2\sin\delta\cos\delta}\left[(K_\lambda - E_\lambda)F'(\delta) - K_\lambda E'(\delta) + \frac{\pi}{2}\right] + K_\lambda \tag{16}$$

where

$$F'(\delta) = \int_0^{\delta} \frac{d\varphi}{\sqrt{1-\lambda'^2\sin^2\varphi}}, \qquad E'(\delta) = \int_0^{\delta} d\varphi\sqrt{1-\lambda'^2\sin^2\varphi} \qquad (\lambda'^2 = 1-\lambda^2)$$

$$n = 1 - \lambda'^2\sin^2\delta$$

Thus, the complete elliptic integral of the third kind J may also be computed directly from tabulated values of the partial and complete elliptic integrals of the first and second kind. The radial velocity component v_{3r} is here continuous at the surface of the vortex cylinder, but has a logarithmic singularity at its edge. The axial velocity component v_{3z} has a discontinuity on the surface of the cylinder, equal to unity, and this was to be expected from our assumption. One half of the discontinuity appears in the expression v_{3z} in explicit form, while the other half is included in the term containing J.

As follows from (10), the values $\partial\psi_2/\partial z$ and $\partial\psi_2/\partial r$ are necessary for the computation of v_{2r} and v_{2z}, because

$$\psi_2 = \frac{r}{4\pi}\int_{z_1}^{z_2}\left[r'\gamma(z')\int_0^{2\pi}\frac{\cos\alpha\, d\alpha}{\rho(r, r', z, \alpha)}\right]dl'$$

therefore

$$4\pi \frac{\partial\psi_2}{\partial r} = \int_{z_1}^{z_2} \left[r'\gamma(z') \int_0^{2\pi} \frac{\cos\alpha d\alpha}{\rho(r, r', z, \alpha)} \right] dl' -$$

$$- r \int_{z_1}^{z_2} \left[r'\gamma(z') \int_0^{2\pi} \frac{(r\cos\alpha - r'\cos^2\alpha)\, d\alpha}{\rho^3(r, r', z, \alpha)} \right] dl' =$$

$$= \frac{2}{\sqrt{\bar{r}}} \int_{z_1}^{z_2} \left\{ \frac{r'\gamma(z')}{\sqrt{k}} \left[(2 - k^2) K_k - 2E_k \right] \right\} dl' -$$

$$- r \int_{z_1}^{z_2} \left[r'\gamma(z') \int_0^{2\pi} \frac{(r\cos\alpha - r'\cos^2\alpha)\, d\alpha}{\rho^3(r, r', z, \alpha)} \right] dl'$$

Here K_k and E_k are complete elliptic integrals of the first and second kind of modulus

$$k = \frac{2\sqrt{rr'}}{\sqrt{(z - z')^2 + (r + r')^2}}$$

Substituting $\alpha = \pi + 2\phi$, it follows

$$\int_0^{2\pi} \frac{(r\cos\alpha - r'\cos^2\alpha)\, d\alpha}{\rho^3(r, r', z, \alpha)} = r \int_{-\pi/2}^{\pi/2} \frac{-2r\cos 2\varphi d\varphi}{\rho^3(r, r', z, \varphi)} - r' \int_{-\pi/2}^{\pi/2} \frac{2\cos^2 2\varphi d\varphi}{\rho^3(r, r', z, \varphi)} =$$

$$\frac{-(r + r')k^4 + 2rk^2 + 8r'k^2 - 8r'}{rr'k(1 - k^2)\sqrt{rr'}} E_k + \frac{4r' - (r + 2r')k^2}{rr'k\sqrt{rr'}} K_k$$

Then

$$v_{2z} = \frac{1}{r}\frac{\partial\psi_2}{\partial r} = \frac{1}{4\pi r} \int_{z_1}^{z_2} \gamma(z') \left[\frac{rk}{\sqrt{rr'}} K_k - \frac{rk(2 - k^2) - r'k^3}{2\sqrt{rr'}(1 - k^2)} E_k \right] dl' =$$

$$= \frac{1}{2\pi} \int_{z_1}^{z_2} \gamma(z') M(z, z')\, dl' \qquad (17)$$

where

$$M(z, z') = \frac{m^2(z, r, z', r') K_k - [(z' - z)^2 + r^2 - r'^2] E_k}{m^2(z, r, z', r')\, m(z, -r, z', r')} \qquad (18)$$

$$m(z, r, z'r') = \sqrt{(z' - z)^2 + (r' - r)^2}$$

Similarly there follows

$$v_{2r} = \frac{1}{2\pi} \int_{z_1}^{z_2} \gamma(z') N(z, z')\, dl' \qquad (19)$$

where

$$\mathring{N}(z, \mathring{z}') = \frac{m^2(z, r, z', r') K_k - [m^2(z, r, z', r') + 2r'r] E_k}{m^2(z, r, z' r')\, m(z, -r, z', r')} \frac{z' - z}{r} \qquad (20)$$

Integrals (17) and (19) are to be solved numerically. But the kernel functions have singularities at $z = z'$ and $r = r'$. Therefore integrals (17) and (19) are transformed in the following way:

$$v_{2z} = \frac{1}{2\pi}\left[\int_{z_1}^{z-\varepsilon} \gamma(z') M(z, z')\, dl' + \int_{z-\varepsilon}^{z+\varepsilon} \gamma(z') M(z, z')\, dl' + \int_{z+\varepsilon}^{z_2} \gamma(z') M(z, z')\, dl'\right] \tag{21}$$

$$v_{2r} = \frac{1}{2\pi}\left[\int_{z_1}^{z-\varepsilon} \gamma(z') N(z, z')\, dl' + \int_{z-\varepsilon}^{z+\varepsilon} \gamma(z') N(z, z')\, dl' + \int_{z+\varepsilon}^{z_2} \gamma(z') N(z, z')\, dl'\right]$$

Results of computations are presented, where expansions in powers of k^1 of the complete elliptic integrals of the first and second kind have been used in the vicinity of $k = 1$

$$\int_{z-\varepsilon}^{z+\varepsilon} \gamma(z') M(z, z')\, dl' = \int_{z-\varepsilon}^{z+\varepsilon} \gamma(z') \frac{m^2(z, r, z', r') K_k - [(z'-z)^2 + r^2 - r'^2] E_k}{m^2(z, r, z', r')\, m(z, -r, z', r')}\, dl' =$$

$$= \int_{z-\varepsilon}^{z+\varepsilon} \frac{\gamma(z')(K_k - E_k)\sqrt{1 + r_z'^2}}{m^2(z, -r, z', r')}\, dz' - 2\int_{z-\varepsilon}^{z+\varepsilon} \frac{\gamma(z') r' (r - r') E_k \sqrt{1 + r_z'^2}\, dz'}{m^2(z, r, z', r')\, m(z, -r, z', r')} \approx$$

$$\approx \frac{\gamma(z)\sqrt{1 + r_z^2}}{r}\, \varepsilon \ln \frac{8r}{\varepsilon\sqrt{1 + r_z^2}} + 2\int_{z-\varepsilon}^{z+\varepsilon} \frac{\gamma(z') r' (r' - r) E_k \sqrt{1 + r_z'^2}\, dz'}{m^2(z, r, z', r')\, m(z, -r, z', r')} \approx$$

$$\approx \frac{\gamma(z)\, \varepsilon \sqrt{1 + r_z^2}}{r} \ln \frac{8r}{\varepsilon\sqrt{1 + r_z^2}} + \frac{\varepsilon\gamma(z) r_z^2}{r\sqrt{1 + r_z^2}} \mp \frac{\pi\gamma(z)}{\sqrt{1 + r_z^2}}$$

Here and in the following pages the minus sign corresponds to the case $r - r' \to +0$, the plus sign corresponds to $r - r' \to -0$; moreover:

$$r_z = dr/dz, \quad r_z' = dr'/dz' \tag{23}$$

Similarly, there is

$$\int_{z-\varepsilon}^{z+\varepsilon} \gamma(z') N(z, z')\, dl' = \frac{1}{r}\int_{z-\varepsilon}^{z+\varepsilon} \gamma(z') \frac{(z'-z)\{m^2(z, r, z', r') K_k - [m^2(z, r, z', r') + 2r'r] E_k\}}{m^2(z, r, z', r')\, m(z, -r, z', r')} \times$$

$$\times \sqrt{1 + r_z'^2}\, dz' = \frac{1}{r}\int_{z-\varepsilon}^{z+\varepsilon} \frac{\gamma(z')(z'-z)(K_k - E_k)\sqrt{1 + r_z'^2}}{m(z, -r', z' r')}\, dz' -$$

$$- 2\int_{z-\varepsilon}^{z+\varepsilon} \frac{\gamma(z') r' (z'-z) E_k \sqrt{1 + r_z'^2}\, dz'}{m^2(z, r, z', r')\, m(z, -r, z', r')} = -\frac{\gamma(z)\, \varepsilon r_z}{r\sqrt{1 + r_z^2}} \mp \frac{\pi\gamma(z) r_z}{\sqrt{1 + r_z^2}} \tag{24}$$

In virtue of (23) and (24) the relations (21) and (22) may be rewritten as

$$v_{2z} = \frac{1}{2\pi}\, v \cdot p \int_{z_1}^{z_2} \gamma(z') M(z, z')\, dl' +$$

$$+\frac{\varepsilon\gamma(z)}{2\pi r}\left[\sqrt{1+r_z^2}\ln\frac{8r}{\varepsilon\sqrt{1+r_z^2}}+\frac{r^2}{\sqrt{1+r_z^2}}\right]\mp\frac{\gamma(z)}{2\sqrt{1+r_z^2}} \quad (25)$$

$$v_{2r}=\frac{1}{2\pi}\ v\cdot p\int_{z_1}^{z_2}\gamma(z')N(z,z')\,dl'-\frac{\gamma(z)\,\varepsilon r_z}{2\pi r\sqrt{1+r_z^2}}\mp\frac{\gamma(z)\,r_z}{2\sqrt{1+r_z^2}} \quad (26)$$

Hence it is evident that v_{2z} and v_{2r} have discontinuities on the surface of the cone and on the curvilinear part of the jet, equal respectively to $\gamma(z)\cos\beta$ and $-\gamma(z)\sin\beta$; here $\gamma(z)=1$ when $z\geqslant 0$, $\beta=\beta_1$ at the surface of the cone, and β is the angle between the tangent to the jet surface and the z axis on the curvilinear part of the jet.

Finally, v_{1z} and v_{1r} are computed. For this $\partial\psi_1/\partial r$ and $\partial\psi_1/\partial z$ are to be determined, as is known from (10). The function ψ_1, allowing for (5) is transcribed

$$\psi_1=\frac{r r_1^2\gamma_1}{4\pi}\int_{-\infty}^{z_1}\left[\frac{1}{r^1}\int_0^{2\pi}\frac{\cos\alpha\,d\alpha}{\rho(r,r',z,\alpha)}\right]dl'$$

then

$$4\pi\frac{\partial\psi_1}{\partial r}=\gamma_1 r_1^2\left\{\int_{-\infty}^{z_1}\left[\frac{1}{r'}\int_0^{2\pi}\frac{\cos\alpha\,d\alpha}{\rho(r,r',z,\alpha)}\right]dl'-\right.$$

$$\left.-r^2\int_{-\infty}^{z_1}\left[\frac{1}{r'}\int_0^{2\pi}\frac{\cos\alpha\,d\alpha}{\rho^3(r,r',z,\alpha)}\right]dl'+r\int_{-\infty}^{z_1}\int_0^{2\pi}\frac{\cos^2\alpha\,d\alpha\,dl'}{\rho^3(r,r',z,\alpha)}\right\}$$

After some transformations and substitution of a variable $r'=1-z'\tan\beta_1$, remembering that

$$\sqrt{1+r_z'^2}=\frac{1}{\cos\beta_1}$$

we obtain

$$4\pi\frac{\partial\psi_1}{\partial r}=\frac{\gamma_1 r r_1^2}{\cos\beta_1}\left\{\frac{r^2-a}{a\operatorname{tg}\beta_1}\int_0^{2\pi}\int_{\infty}^{r_1}\frac{d\alpha\,dr'}{(cr'^2+br'+a)^{3/2}}-\frac{r^2}{a\operatorname{tg}\beta_1}\int_0^{2\pi}\int_{\infty}^{r_1}\frac{\cos^2\alpha\,d\alpha\,dr'}{(cr'^2+br'+a)^{3/2}}+\right.$$

$$\left.+\frac{2r}{a\operatorname{tg}\beta_1}\int_0^{2\pi}\left(\frac{b\sqrt{c}}{\Delta}-\frac{bcr_1-2ac+b^2}{\Delta\sqrt{cr_1^2+br_1+a}}\right)\cos\alpha\,d\alpha\right\} \quad (27)$$

where

$$a=r^2+(z-\operatorname{ctg}\beta_1)^2,\quad b=2(z-\operatorname{ctg}\beta_1-\operatorname{ctg}^2\beta_1-r\cos\alpha),$$

$$r_1=1-z_1\operatorname{tg}\beta_1,\ c=1+\operatorname{ctg}^2\beta_1$$

$$\Delta=-4\left[r^2\cos^2\alpha+2r\operatorname{ctg}\beta_1(\operatorname{ctg}\beta_1-z)\cos\alpha-(1+\operatorname{ctg}^2\beta_1)r^2-(\operatorname{ctg}\beta_1-z)^2\right]$$

All integrals included in (27) after some transformations may be resolved into elliptic integrals of the first, second and third kind.

$$4\pi\frac{\partial\psi_1}{\partial r}=\frac{\gamma_1 r r_1{}^2\sigma}{a\sqrt{rr_1}\sin\beta_1}\left[An_1J_1+Bn_2J_2+DK_\sigma-\frac{4r}{\sigma^2}E_\sigma\right]-$$

$$-\frac{\gamma_1 r_1{}^2\sqrt{c}\,\pi}{ar\sin\beta_1}\left[\frac{(2a-2r^2-b_1x_1)}{(x_2-x_1)\sqrt{x_1{}^2-1}}+\frac{(2a-2r^2-b_1x_2)}{(x_2-x_1)\sqrt{x_2{}^2-1}}\right] \quad (28)$$

Here

$$\sigma=\frac{2\sqrt{rr_1}}{\sqrt{(z-z_1)^2+(r+r_1)^2}}$$

$$x_1=\frac{-\operatorname{ctg}\beta_1(\operatorname{ctg}\beta_1-z)+\sqrt{c}\sqrt{(\operatorname{ctg}\beta_1-z)^2+r^2}}{r}$$

$$x_2=-\frac{\operatorname{ctg}\beta_1(\operatorname{ctg}\beta_1-z)+\sqrt{c}\sqrt{(\operatorname{ctg}\beta_1-z)^2+r^2}}{r}$$

$$D=2\left[\operatorname{ctg}\beta_1(\operatorname{ctg}\beta_1-z)-r+\frac{2r}{\sigma^2}\right]$$

$$A=\frac{-(\operatorname{ctg}\beta_1-z)\sqrt{c}}{2r\sqrt{(\operatorname{ctg}\beta_1-z)^2+r^2}}\{\operatorname{ctg}\beta_1(\operatorname{ctg}\beta_1-z)^2+r_1c(\operatorname{ctg}\beta_1-z)+$$
$$+r^2\operatorname{ctg}\beta_1-\sqrt{c}\,[\operatorname{ctg}\beta_1-z+r_1\operatorname{ctg}\beta_1]\sqrt{r^2+(\operatorname{ctg}\beta_1-z)^2}\}$$

$$B=\frac{(\operatorname{ctg}\beta_1-z)\sqrt{c}}{2r\sqrt{(\operatorname{ctg}\beta_1-z)^2+r^2}}\{\operatorname{ctg}\beta_1(\operatorname{ctg}\beta_1-z)^2+r_1c(\operatorname{ctg}\beta_1-z)+r^2\operatorname{ctg}\beta_1+$$
$$+\sqrt{c}\,[\operatorname{ctg}\beta_1-z+r_1\operatorname{ctg}\beta_1]\sqrt{r^2+(\operatorname{ctg}\beta_1-z)^2}\}$$

$$n_1=\frac{2r}{r-\operatorname{ctg}\beta_1(\operatorname{ctg}\beta_1-z)+\sqrt{c}\sqrt{(\operatorname{ctg}\beta-z)^2+r^2}}$$

$$n_2=\frac{2r}{r-\operatorname{ctg}\beta_1(\operatorname{ctg}\beta_1-z)-\sqrt{c}\sqrt{(\operatorname{ctg}\beta-z)^2+r^2}}$$

$$b_1=2r\operatorname{ctg}\beta_1(\operatorname{ctg}\beta_1-z)$$

Here J_1 and J_2 are the complete elliptic integrals of the third kind, that is

$$J_1=J_1(-n_1,\sigma)=\int_0^{\frac{1}{2}\pi}\frac{d\varphi}{(1-n_1\sin^2\varphi)\sqrt{1-\sigma^2\sin^2\varphi}}$$

$$J_2=J_2(-n_2,\sigma)=\int_0^{\frac{1}{2}\pi}\frac{d\varphi}{(1-n_2\sin^2\varphi)\sqrt{1-\sigma^2\sin^2\varphi}}$$

K_σ and E_σ are complete elliptic integrals of the first and second kind of modulus σ, see (13).

Two cases are to be distinguished. Results for the function v_{1r} are developed simultaneously.

First case: $\cot\beta_1 > z$

$$v_{1z} = \frac{\gamma_1 r_1^2 \sigma}{4\pi\sqrt{rr_1 a}\sin\beta^1}\Big[(\operatorname{ctg}\beta_1 - z)\,p(z, r)J_1 + (\operatorname{ctg}\beta_1 - z)\,Q(z, r)J_2 +$$
$$+ D(z, r)K_\sigma - \frac{4r}{\sigma^2}E_\sigma\Big] + \frac{\gamma_1 r^2(\operatorname{ctg}\beta_1 - z)}{4\sin^2\beta_1 a^{3/2}} \mp \frac{\gamma_1 r_1^2(\operatorname{ctg}\beta_1 - z)}{4\sin^2\beta_1 a^{3/2}} \qquad (29)$$

$$v_{1r} = -\frac{\gamma_1 r_1^2 \sigma}{4\pi\sqrt{rr_1 a}\sin\beta_1}\Big[r\rho(z, r)J_1 + rQ(z, r)J_2 + \qquad (30)$$
$$+ D_1^*(z, r)K_\sigma - \frac{4(z - \operatorname{ctg}\beta_1)}{\sigma^2}E_\sigma\Big] - \frac{\gamma_1 r_1^2 r}{4\sin^2\beta_1 a^{3/2}} \pm \frac{\gamma_1 r_1^2 r}{4a^{3/2}\sin^2\beta_1}.$$

The minus sign in the formula (29) corresponds to $|(1 - r)/z| > |\tan\beta_1|$, the plus sign to $|(1 - r)/z| < |\tan\beta_1|$, but the opposite holds good in formula (30).

Second case: $\cot\beta_1 < z$

$$v_{1z} = \frac{\gamma_1 r_1^2 \sigma}{4\pi\sqrt{rr_1 a}\sin\beta_1}\Big[(\operatorname{ctg}\beta_1 - z)\rho(z, r)J_1 + (\operatorname{ctg}\beta_1 - z)Q(z, r)J_2 +$$
$$+ D(z, r)K_\sigma - \frac{4r}{\sigma^2}E_\sigma\Big] + \frac{\gamma_1 r_1^2(z - \operatorname{ctg}\beta_1)}{4a^{3/2}\sin^2\beta_1} \mp \frac{\gamma_1 r_1^2(z - \operatorname{ctg}\beta_1)}{4a^{3/2}\sin^2\beta_1} \qquad (31)$$

$$v_{1r} = -\frac{\gamma_1 r_1^2 \sigma}{4\pi\sqrt{rr_1 a}\sin\beta_1}\Big[r\rho(z, r)J_1 + rQ(z, r)J_2 + \qquad (32)$$
$$+ D_1^*(z, r)K_\sigma - \frac{4(z - \operatorname{ctg}\beta_1)}{\sigma^2}E_\sigma\Big] + \frac{\gamma_1 r_1^2 r}{4a^{3/2}\sin\beta_1} \mp \frac{\gamma_1 r_1^2 r}{4a^{3/2}\sin^2\beta_1}$$

The minus sign in formulas (31) and (32) corresponds to $|(r + 1)/z| > |\tan\beta_1|$, the plus sign to $|(r + 1)/z| < |\tan\beta_1|$, thus

$$p(z, r) = \frac{-\sqrt{c}\,\{cr_1(\operatorname{ctg}\beta_1 - z) + a\operatorname{ctg}\beta_1 - [(\operatorname{ctg}\beta_1 - z) + r_1\operatorname{ctg}\beta_1]\sqrt{ac}\}}{\sqrt{ca} + [r - \operatorname{ctg}\beta_1(\operatorname{ctg}\beta_1 - z)]\sqrt{a}}$$

$$Q(z, r) = \frac{\sqrt{c}\,\{cr_1(\operatorname{ctg}\beta_1 - z) + a\operatorname{ctg}\beta_1 + [(\operatorname{ctg}\beta_1 - z) + r_1\operatorname{ctg}\beta_1]\sqrt{ac}\}}{-\sqrt{ca} + [r - \operatorname{ctg}\beta_1(\operatorname{ctg}\beta_1 - z)]\sqrt{a}}$$

$$D_1^*(z, r) = 2\Big[r\operatorname{ctg}\beta_1 - (z - \operatorname{ctg}\beta_1) + \frac{2(z - \operatorname{ctg}\beta_1)}{\sigma^2}\Big]$$

Complete elliptic integrals of the third kind resolve into elliptic integrals of the first and second kind.

It is easy to prove that $\sigma^2 \leqslant n_1$, therefore J_1 will be (4) after an auxiliary angle δ_1 is introduced by the equation $n_1 = 1 - \sigma'^2\sin^2\delta_1$:

$$J_1(-n_1\,\sigma) = \frac{\sqrt{1 - \sigma'^2\sin^2\delta_1}}{\sigma'^2\sin\delta_1\cos\delta_1}\Big[(K_\sigma - E_\sigma)F'(\delta_1) - K_\sigma E'(\delta_1) + \frac{\pi}{2}\Big] + K_\sigma$$

Previous notations are retained here.

The integral $J_2(-n_2, \sigma)$ will be also expressed in terms of elliptic integrals of the first and second kind. In this case an auxiliary angle

δ_2 is introduced by the equation

$$-n_2 = \operatorname{ctg}^2 \delta_2 \, (n_2 < 0)$$

We obtain

$$J_2(-n_2, \sigma) = K_\sigma \sin^2\delta_2 + \frac{\sin\delta_2\cos\delta_2}{\sqrt{1-\sigma'^2\sin^2\delta_2}}\left[(K_\sigma - E_\sigma) F'(\delta_2) - K_\sigma E(\delta_2) + \frac{\pi}{2}\right]$$

This formula was proposed by Frankl'. Formulas for J_1 and J_2 can be checked by a solution for the values in parentheses containing partial elliptical integrals, and by differentiation by the upper limit; see also (5).

The singularities of the function v_{1r} and v_{1z} are to be considered.

1. When $\cot \beta_1 > z$. These are two possibilities: a) $z < z_1$, b) $z > z_1$.

The extreme values of axial and radial velocity components at the internal and external sides of the cone are designated by $v_{1z}^{(1)}$, $v_{1z}^{(2)}$ and $v_{1r}^{(1)}$, $v_{1r}^{(2)}$. Then, for the case $z < z_1$

$$v_{1z}^{(1)} - v_{1z}^{(2)} = \frac{\gamma_1 r_1^2}{r^2}\cos\beta_1, \qquad v_{1r}^{(1)} - v_{1r}^{(2)} = -\frac{\gamma_1 r_1^2}{r^2}\sin\beta_1$$

Otherwise, the tangent of the velocity component has a discontinuity $\gamma_1 r_1^2/r^2$, on the surface of the cone (for $z < z_1$), which was to be expected from our assumption

In the case $z > z_1$, this velocity has no discontinuity, since the member containing J_2 in formulas (29) and (30) gives discontinuities equal in magnitude and opposite in signs to those in explicit form in formulas (29) and (30), and they therefore cancel out.

2. When $\cot \beta_1 < z$, the tangent of the velocity component has no discontinuity, as in the case $z > z_1$.

Hence, v_z and v_r are completely investigated and computed.

Substitution of values v_z and v_r into (4) gives the integrodifferential equation for $\gamma(z)$ and $r(z)$:

$$v_{1r} + v_{2r} + v_{3r} - (v_{1z} + v_{2z} + v_{3z})\frac{dr}{dz} = 0 \tag{33}$$

The solution of the integrodifferential equation determines the form of the jet and the velocity distribution along the solid wall.

Equation (33) is to be solved approximately.

Computations have been performed below for the case $\beta = 1/4\,\pi$. For the first approximation to the solution of the equation (33) the form of the jet has been taken from the plane problem (1). In this case equation (33) is reduced to the integral equation for $\gamma(z)$. After $\gamma(z)$ is determined, the form of the jet is computed by a second approximation, and so on.

If $2b$ is the width of the jet in infinity downstream in the two-dimensional problem, the radius of the jet at infinity in a three-dimensional problem has a value $r_0 = \sqrt{b}$. Then a cylinder of radius r_0 is drawn from infinity up to the curvilinear part of the jet in the two-dimensional problem, and this shape of the jet is taken as the first approximation. The method of solution of equation (33) is as follows: $dr/dz = -1$ when $\beta = 1/4\,\pi$, therefore from (33)

$$v_{1r} + v_{2r} + v_{3r} + v_{1z} + v_{2z} + v_{3z} = 0 \tag{34}$$

Here all terms, except v_{2r} and v_{2z}, are expressed in terms of complete elliptic integrals of the first, second and third kind, while v_{2r} and v_{2z} contain integrals solved numerically at the points

$$z = -0.1,\ -0.3,\ -0.5,\ -0.7,\ -0.9$$

The integral equation (34) is solved, as usual, by a system of linear algebraic equations:

$$\begin{aligned}
&1.\ 917\ \gamma_1 + 0.591\ \gamma_2 + 0.804\ \gamma_3 + 1.264\ \gamma_4 + 1.993\ \gamma_5 = 1.361\\
&2.\ 241\ \gamma_1 + 0.790\ \gamma_2 + 1.232\ \gamma_3 + 1.923\ \gamma_4 - 0.879\ \gamma_5 = 0.817\\
&2.\ 691\ \gamma_1 + 1.215\ \gamma_2 + 1.870\ \gamma_3 - 0.939\ \gamma_4 - 0.583\ \gamma_5 = 0.437\\
&3.\ 444\ \gamma_1 - 1.828\ \gamma_2 - 0.984\ \gamma_3 - 0.618\ \gamma_4 - 0.272\ \gamma_5 = 0.247\\
&4.\ 236\ \gamma_1 - 1.022\ \gamma_2 - 0.646\ \gamma_3 - 0.297\ \gamma_4 - 0.155\ \gamma_5 = 0.168
\end{aligned}$$

Approximate solution of this system gives:

$$\gamma_5 = \gamma(-0.2) \approx 0.296,\quad \gamma_4 = \gamma(-0{,}4) \approx 0.243,\quad \gamma_3 = \gamma(-0{,}6) \approx 0{,}184,$$
$$\gamma_2 = \gamma(-0.8) \approx 0.136 = \gamma_1 = \gamma(-1) \approx 0.119$$

For the form of the jet we have taken as a first approximation:

$z = 0.0$	0.1	0.2	0.3	0.4	0.5
$r(z) = 1$	0.938	0.8957	0.8690	0.8645	0.8645

$\gamma(z)$ for this form of the jet has been computed:

$z =$	-0.2	-0.4	-0.6	-0.8	-1.0
$\gamma(z) =$	0.296;	0.243;	0.184;	0.134;	0.119;

A relationship $dr/dz = v_r/v_z$ has been used for the second approximation of the form of the jet:

$z =$	0.05,	0.15,	0.25	0.35,	0.45
$dr/dz =$	-0.4953,	-0.3250,	-0.2160,	-0.1794,	-0.0338

therefore from

$$r = 1 + \int_0^z \left(\frac{dr}{dz}\right) dz$$

the values are

$$z = 0.05, \quad 0.15, \quad 0.25, \quad 0.35, \quad 0.45$$
$$r(z) = 0.9626, \quad 0.9266, \quad 0.8996, \quad 0.8798, \quad 0.8691$$

Thus, a discharge coefficient of about 0.75 has been found from the second approximation.

BIBLIOGRAPHY

1. Zhukovskii, N.E., *Vidoizmenenie metoda Kirchhoffa dlia opredeleniia dvizheniia zhidkosti v dvukh izmereniiakh pri postoiannoi skorosti dannoi na neizvestnoi linii toka (Modification of the Kirchhoff method for determination of the motion of a liquid in two dimensions when constant velocity is given along an unknown streamline.* Complete works, Vol. 3, ONTI NKTP, Moscow-Leningrad, 1936.

2. Trefftz, E., Uber die Kontraktionen kreisformiger Flussigkeitsstrahlen. *Z. angew. Math. und Phys.* Vol. 64, No. 1, 1916.

3. Kochin, N.E., Roze, N.V. and Kibel', A.I., *Teoreticheskaia gidromekhanika (Theoretical hydromechanics)*. Vol. 1, 1955.

4. Zhukovskii, N.E., *Polnoe sobranie sochinenii (Complete works)*. Vol.6, 1937 [see appendix by V.P. Vetchinkin to the article *Vortex theory of the ship propeller*, article 1, p. 138] .

5. Tricomi, F., *Elliptic functions and integrals* (In a book by several authors *Higher transcendental functions*, Vol. 2, New York-Toronto-London, 1953).

6. Segal, B.I. and Semendiaev, K.A., *Mathematical tables with five decimals*. AN SSSR, 1950.

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